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Effective action method for the Langevin equation

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A general approach to nonlinear stochastic equations with white noise is proposed. It consists of a path integral representation of the nonlinear Langevin equation and allows for systematic approximations. The present method is not restricted to the asymptotic, i.e., stationary, regime and is suited for deriving equations describing the relaxation of a system from arbitrary initial conditions. After reducing the nonlinear Langevin equation to an equivalent equilibrium problem for the generating functional, we are able to apply known techniques of conventional equilibrium statistical field theory. We extend the effective action method developed in quantum field theory by Cornwall, Jackiw, and Tomboulis [Phys. Rev. D **10**, 2428 (1974)] to nonequilibrium processes. Arguments are given as to its superiority over perturbative schemes. These are illustrated by studying an N-component Ginzburg-Landau equation in zero spatial dimension in the limit of large N. Within this limit we show the equivalence of the lowest order approximation, i.e., the dressed loop expansion, with the Gaussian variational ansatz for the effective potential, which leads to the dynamical Hartree approximation.

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I. INTRODUCTION

Nonlinear stochastic differential equations (SDEs) in one or more variables occur frequently in the description of a wide variety of physical phenomena such as those involving relaxation towards a steady state [1,2]. A celebrated example of a SDE is the Langevin equation, where one relates the rate of change of some physical observables to a drift term, i.e., to a deterministic driving force, plus a stochastic noise. Whereas stochastic linear differential equations are in principle amenable to analytic solutions, those of the nonlinear type are much more difficult to treat and in most cases one has to resort either to computer simulations or to approximate schemes. Several approximations exist in the literature for the analysis of nonlinear SDEs. A widely used method is the so-called statistical linearization [3], which consists of the replacement of the nonlinear SDEs by equivalent linear ones whose coefficients are determined by a suitable error minimization algorithm. This method, in its original form, is appropriate in the asymptotic, i.e., stationary, regime. In many physical situations, however, one is interested in the transient regime. To deal with this problem the idea of statistical linearization has been extended to include a possible time dependence in the parameters [4,5]. The drawback of these methods is that they are not derived in a systematic way, so it is not simple to improve them.

An alternative procedure is the so-called dynamical Hartree approximation. This scheme amounts to neglecting cumulants of order higher than the second and is exact whenever the probability distribution associated with the problem is Gaussian. This is not the case of strongly interacting systems or in the presence of large fluctuations, so corrections to the Hartree approximation should be taken into account. Also in this case it is not simple to improve the approximation.

In this paper we present a formulation of the nonlinear SDE that allows for systematic approximations. This is achieved by reducing the nonlinear Langevin equation to an equivalent equilibrium problem, which can be analyzed with the methods of conventional equilibrium statistical field theory. In particular we have applied a method originally developed in quantum field theory by Cornwall, Jackiw, and Tomboulis [6,7], alternative to conventional perturbation theory, because a standard coupling constant expansion can only be used for the study of small corrections to the deterministic result. In this respect the present approach is alternative to the field theoretical treatment based on the introduction of auxiliary fields; see, e.g., [8].

To illustrate the method we shall study an N-component Ginzburg-Landau equation in zero spatial dimension with the purpose of deriving systematically the time dependent Hartree equations and the first corrections. The same model has been discussed previously by Bhattacharjee, Meakin, and Scalapino [9], who introduced an approximation scheme for the Langevin equation based on leading terms of the expansion in the small parameter $1/N$. We shall obtain a solution that represents a systematic expansion in $1/N$. This will be discussed in Secs. III–V, where the corrections to the $N \to \infty$ solution are considered. The formalism is established in Sec. II, where we construct the generating functional for the average value of the observables and its correlations.

The path integral representation of the generating functional used here is well suited for a variational approach similar to the Feynman method in equilibrium statistical mechanics. This is presented in Sec. VI. The variational approach provides an alternative and elegant derivation of the dynamical Hartree equations. This method is quite useful when it is difficult to identify a small parameter for the expansion, as, for example, in the model discussed in Sec. VI for the limiting case $N = 1$. We stress, however, that the variational approach gives only an approximation and in general it is difficult to improve it systematically.

II. FORMALISM

In this section we derive the effective action formalism for the Langevin equation. The path integral method constitutes a convenient representation of the Langevin equation for a field $\phi(t)$. Within this approach, the original stochastic differential equation, where $\phi(t)$ depends on another field $\xi(t)$, called the noise, is reformulated by constructing an effective action for the field ϕ obtained by integrating out the noise. The advantage of this transformation is that one can employ the well-known methods of equilibrium statistical field theory. To keep the notation as simple as possible, the derivation will be carried out for a single component real field. The extension to N-vector fields will be discussed later.

The time evolution of the field $\phi(t)$ is governed by the Langevin equation

$$\frac{\partial \phi}{\partial t} = -\frac{\partial}{\partial \phi} S[\phi] + \xi, \tag{1}$$

where $S[\phi]$ is an "energy" function and ξ a Gaussian random variable satisfying the following properties:

$$\langle \xi(t) \rangle = 0, \qquad \langle \xi(t)\, \xi(t') \rangle = \Gamma\, \delta(t - t'). \tag{2}$$

Our aim is to study the correlations associated with the stochastic process (1). Instead of working directly with Eq. (1), we find it convenient to construct a generating functional from which the correlations can be obtained. Proceeding in the standard way (see, e.g., Ref. [10]), we introduce an external time-dependent source $J(t)$ and define the generating functional

$$Z[J] = \mathcal{N} \int \mathcal{D}''\phi\, \mathcal{D}\xi\, \mathcal{P}(\phi(0))\, \delta(\phi - \phi_\xi) \times \exp\left[-\int_0^\tau dt\, J\phi\right] \exp\left[-\int_0^\tau dt\, \frac{\xi^2}{2\Gamma}\right], \tag{3}$$

where ϕ_ξ is the solution of stochastic equation (1), for a given realization of noise $\xi(t)$, subject to some set of initial value conditions $\phi(0)$ assigned with probability $\mathcal{P}(\phi(0))$, and $\mathcal{N}$ is a normalizing constant. The functional integral on ϕ in Eq. (3) includes integration over $\phi(0)$ and $\phi(\tau)$. We denote it by the double prime: $\mathcal{D}''\phi$. The time τ is an arbitrary time which for convenience can be eventually assumed infinite. The δ function stands for

$$\delta(\phi - \phi_\xi) = \delta\left[\frac{\partial \phi}{\partial t} + \frac{\partial S}{\partial \phi} - \xi\right] \det\left|\frac{\delta \xi}{\delta \phi}\right|, \tag{4}$$

where the factor $\det|\delta\xi/\delta\phi|$ represents the Jacobian of the transformation $\xi \to \phi$. With well-known manipulations (see, e.g., Refs. [10,11]), one obtains

$$\det\left|\frac{\delta \xi}{\delta \phi}\right| = \exp\left[\frac{1}{2}\int_0^\tau dt\, \frac{\partial^2 S}{\partial \phi^2}\right]. \tag{5}$$

In deriving Eq. (5) we have used the forward time propagation Green function $\theta(t - t')$ of the operator ∂_t and the definition

$$\theta(0) = \frac{1}{2}. \tag{6}$$

This choice corresponds to the "physical" regularization of the noise term ξ as

$$\langle \xi(t)\, \xi(t') \rangle = \Gamma\, \eta(t - t'), \tag{7}$$

where $\eta(t)$ is an even function sharply peaked at $t = 0$, whose integral from $-\infty$ to $+\infty$ is equal to 1. The δ-function-correlated noise is obtained in the limit of vanishing width. In terms of stochastic differential equations this corresponds to the Stratonovich formalism [12].

At this stage one eliminates the noise field by inserting Eq. (5) into Eq. (3) and performing the integral over the noise ξ obtaining

$$Z[J] = \mathcal{N} \int \mathcal{D}''\phi\, \mathcal{P}(\phi(0)) \times \exp\left[-\int_0^\tau dt\, \frac{1}{2\Gamma}\left(\frac{\partial \phi}{\partial t} + \frac{\partial S}{\partial \phi}\right)^2 + \frac{1}{2}\int_0^\tau dt\, \frac{\partial^2 S}{\partial \phi^2} - \int_0^\tau dt\, J\phi\right]. \tag{8}$$

The argument of the exponential can be simplified by performing the integration of the term $\int_0^\tau dt\, \dot{\phi}\, \partial S/\partial\phi = S[\phi(\tau)] - S[\phi(0)]$, so we finally have

$$Z[J] = \mathcal{N} \int \mathcal{D}\phi(0)\, \mathcal{P}(\phi(0))\, e^{S[\phi(0)]/\Gamma} \mathcal{D}\phi(\tau)\, e^{S[\phi(\tau)]/\Gamma} \times \mathcal{D}\phi\, \exp\left[-I[\phi] - \int_0^\tau dt\, J\phi\right], \tag{9}$$

where $\mathcal{D}\phi$ denotes integration over all paths starting at $\phi(0)$ for $t = 0$ and ending at $\phi(\tau)$ for $t = \tau$. It is defined as

$$\mathcal{D}\phi = \lim_{N\to\infty} \prod_{i=1}^{N-1} d\phi(t_i), \tag{10}$$

where $\phi(t_i)$ is the field at time $t_i = i\epsilon$, having sliced the interval 0 to τ in N parts of size $\epsilon = \tau/N$. The action $I(\phi)$ is given by

$$I[\phi] = \frac{1}{2\Gamma}\int_0^\tau dt\, \left[\left(\frac{\partial \phi}{\partial t}\right)^2 + \left(\frac{\partial S}{\partial \phi}\right)^2\right] - \frac{1}{2}\int_0^\tau dt\, \frac{\partial^2 S}{\partial \phi^2} \tag{11}$$

and contains coupling terms of different structure from the original energy $S[\phi]$. Since in Eq. (9) the integration over the end points only fixes the boundary conditions at $t = 0$ and $t = \tau$, without lost of generality we can consider a "reduced" generating functional

$$Z[J] = \mathcal{N} \int_{0,\phi_0}^{\tau,\phi_1} \mathcal{D}\phi \, \exp\left[-I[\phi] - \int_0^\tau dt\, J\phi\right], \quad (12)$$

where the functional integral contains all paths that start at $t = 0$ from $\phi(0) = \phi_0$ and end at $\phi(\tau) = \phi_1$ for $t = \tau$. In the limit $\tau \to \infty$ the paths become independent of the final value $\phi(\tau)$.

We note, however, that the presence of the additional constraint is necessary to select paths that are solutions of the original equation of motion Eq. (1). Equations (11) and (12) lead in fact to second-order differential equations of motion, whereas the original stochastic equation is of the first order. The additional constraint at $t = \tau$ makes the problem well defined since the paths in Eq. (12) must satisfy the two constraints $\phi(0) = \phi_0$ and $\phi(\tau) = \phi_1$. Once the two boundary conditions are imposed, the path is also solution of the first-order differential equation (1), as can be easily seen in the limit $\Gamma \to 0$, i.e., the deterministic limit.

In general the calculation of path integrals such as Eq. (12) is not at all straightforward. Nevertheless, quantities of physical interest can be obtained. In our case we are interested in the noise-averaged value of the field $\langle\phi(t)\rangle$ and correlations $\langle\phi(t)\,\phi(t')\rangle$ as functions of time. An advantage of the present formalism is that self-consistent, systematic variational principles for these quantities can be obtained using a method introduced by Cornwall, Jackiw, and Tomboulis in quantum field theory [6]. The basic idea is to derive an effective action that is stationary at the physical values of $\langle\phi(t)\rangle$ and $\langle\phi(t)\,\phi(t')\rangle$.

The method starts by generalizing Eq. (12) to account for the composite operator $\phi(t)\phi(t')$. We then define the generating functional

$$Z[J,K] = \mathcal{N} \int_{0,\phi_0}^{\tau,\phi_1} \mathcal{D}\phi \, \exp\Bigg[-I[\phi] - \int_0^\tau dt\, J(t)\,\phi(t) -\frac{1}{2}\int_0^\tau dt \int_0^\tau dt'\, \phi(t)\, K(t,t')\phi(t')\Bigg], \quad (13)$$

where J and K are a local and a bilocal source, respectively.

By taking functional derivatives with respect to the external sources, the averaged correlations of ϕ can be obtained. In particular, by considering $W[J,K] = -\ln Z[J,K]$, we have for $0 < s, s' < \tau$,

$$\frac{\delta}{\delta J(s)}\, W[J,K] = \langle\phi(s)\rangle \equiv q(s),$$
$$\frac{\delta}{\delta K(s,s')}\, W[J,K] = \tfrac{1}{2}\langle\phi(s)\,\phi(s')\rangle \equiv \tfrac{1}{2}\left[q(s)\,q(s') + G(s,s')\right], \quad (14)$$

where the averages are obtained with the weight of Eq. (13). In the limit of vanishing external sources, q and G become the noise-averaged field $\langle\phi(s)\rangle$ and the connected two-point correlation function $\langle\phi(s)\,\phi(s')\rangle_c$ of the process described by the Langevin equation (1).

By Legendre transforming $W[J,K]$ we can eliminate J and K in favor of q and G:

$$\Gamma[q,G] = W[J,K] - \int_0^\tau ds\, q(s)\, J(s) -\frac{1}{2}\int_0^\tau ds \int_0^\tau ds'\, q(s)\, K(s,s')\, q(s') -\frac{1}{2}\int_0^\tau ds \int_0^\tau ds'\, G(s,s')\, K(s,s'), \quad (15)$$

where J and K are eliminated as a function of q and G by the use of Eq. (14). It can be shown that $\Gamma[q,G]$ is the generating function of two-particle irreducible (2PI) Green functions, i.e., it is given by all diagrams that cannot be separated in two pieces by cutting two lines [6,7]. The external sources can be obtained from $\Gamma[q,G]$ as

$$\frac{\delta}{\delta q(s)}\,\Gamma[q,G] = -J(s) - \int_0^\tau ds\, K(s,s')\, q(s),$$
$$\frac{\delta}{\delta G(s,s')}\,\Gamma[q,G] = -\frac{1}{2}\,K(s,s'). \quad (16)$$

The physical process corresponds to vanishing sources $J = K = 0$. From Eq. (16) it follows that in this limit the value of q and G are determined by the stationary point of $\Gamma[q,G]$. We have thus obtained a variational principle for the noise-averaged field $\langle\phi(s)\rangle$ and the connected two-point correlation function $\langle\phi(s)\,\phi(s')\rangle_c$ of the process described by the Langevin equation (1).

The next step is to evaluate $\Gamma[q,G]$. Following Refs. [6,7] $\Gamma[q,G]$ can be written as

$$\Gamma[q,G] = I[q] + \frac{1}{2}\,\mathrm{Tr}\,\ln G^{-1} + \frac{1}{2}\,\mathrm{Tr}\,\mathcal{D}^{-1}[q]\, G + \Gamma_2[q,G] + \mathrm{const}, \quad (17)$$

where $I[q]$ is given by Eq. (11) with $\phi \to q$,

$$\mathcal{D}^{-1}[q] \equiv \left.\frac{\delta^2 I[\phi]}{\delta\phi(s)\delta\phi(s')}\right|_{\phi=q} = D^{-1} + \left.\frac{\delta^2 I_{\rm int}[\phi;q]}{\delta\phi(s)\delta\phi(s')}\right|_{\phi=q} \quad (18)$$

with D^{-1} propagator of the "free" theory. The functional Γ_2 is given by the sum of all 2PI vacuum diagrams of a theory with interactions determined by $I_{\rm int}$ and propagators G. The interaction term is defined by the shifted action

$$I[q+\phi] - I[q] - \phi\left.\frac{\delta I[\phi]}{\delta\phi}\right|_{\phi=q} = \frac{1}{2}\phi\mathcal{D}^{-1}[q]\phi + I_{\rm int}[\phi;q]. \quad (19)$$

This procedure corresponds to a dressed loop expansion with vertices that depend on ϕ and can thus exhibit nonperturbative effects even for a small number of dressed loops. The crucial point is that it in no sense corresponds to a perturbation theory in physical amplitudes. The stationarity conditions for $\Gamma[q,G]$ yield a coupled set of nonlinear dynamical equations for ϕ and G. If one could sum up the whole series, the exact value of ϕ and G would emerge from the stationary point. If the series is truncated one gets approximate values of ϕ and G.

However, the resulting equations describe nonperturbative behaviors.

In the next section we apply the above formalism to a O(N) problem where the successive contributions to Γ_2 can be extracted in the $N \to \infty$ limit. The leading order reproduces the results obtained with other approaches such as, e.g., the statistical linearization. Next orders give systematic corrections.

III. MODEL

To illustrate the formalism introduced in the preceding section we consider an N-component Ginzburg-Landau time-dependent field ϕ_i with quadratic local interaction in zero spatial dimension. When discussing fluctuations effects to any given order in a perturbation expansion one is not usually able to justify the neglect of yet higher orders. However, for theories with large N internal symmetry group there exists another perturbative scheme, the $1/N$ expansion. The model is specified by the evolution equation

$$\frac{\partial \phi_i}{\partial t} = -\frac{\partial}{\partial \phi_i} S[\boldsymbol{\phi}] + \xi_i, \tag{20}$$

$$S[\boldsymbol{\phi}] = \frac{a}{2}\boldsymbol{\phi}^2 + \frac{\lambda}{4!N}(\boldsymbol{\phi}^2)^2, \tag{21}$$

where Gaussian random noise is defined by

$$\langle \xi_i(t)\rangle = 0, \qquad \langle \xi_i(t)\,\xi_j(t')\rangle = \Gamma\,\delta_{ij}\delta(t-t') \tag{22}$$

and we assume that $a < 0$ and $\lambda > 0$. Generalizing Eq. (11) to N-component fields one finds the action

$$I[\boldsymbol{\phi}] = \frac{1}{\Gamma}\int_0^\tau dt \left[\frac{\dot{\boldsymbol{\phi}}^2}{2} + \frac{\widetilde{m}^2}{2}\boldsymbol{\phi}^2 + \frac{\lambda_0}{4!N}(\boldsymbol{\phi}^2)^2 + \frac{g_0}{6!N^2}(\boldsymbol{\phi}^2)^3 - \frac{aN\Gamma}{2}\right], \tag{23}$$

where the parameters $\widetilde{m}^2$, λ_0, and g_0 are related to the original constants by

$$\widetilde{m}^2 = m^2 + \delta m^2 = a^2 - \frac{\lambda\Gamma}{6} - \frac{\lambda\Gamma}{3N}, \tag{24a}$$

$$\delta m^2 = -\frac{\lambda\Gamma}{3N}, \tag{24b}$$

$$\lambda_0 = 4\,a\lambda, \tag{24c}$$

$$g_0 = 10\,\lambda^2. \tag{24d}$$

The last term in Eq. (23) does not depend on ϕ and can be absorbed into the definition of the normalizing constant $\mathcal{N}$ in Eq. (12).

In the limit $N \to \infty$ we can calculate explicitly the leading order term of the functional (17) following the same steps of Ref. [13]. The "action" (23) corresponds to a classical ϕ^6 theory in one spatial dimension.

From Eq. (23) it follows that the leading contributions to Eq. (17) for $N \to \infty$ are

$$\frac{1}{2}\,\mathrm{Tr}\,\mathcal{D}^{-1}G = \frac{N}{2\Gamma}\int_0^\tau dt\int_0^\tau dt'\left[-\frac{\partial^2}{\partial t^2} + m^2 + \frac{\lambda_0}{3!N}q^2(t) + \frac{g_0}{5!N^2}q^4(t)\right] G(t,t')\,\delta(t-t') \tag{25}$$

and

$$\mathcal{D}^{-1}(t,t') = \frac{1}{\Gamma}\left[-\frac{\partial^2}{\partial t^2} + m^2 + \frac{\lambda_0}{3!N}q^2(t) + \frac{g_0}{5!N^2}q^4(t)\right]\delta(t-t'). \tag{26}$$

The leading order 2PI diagrams $N \to \infty$, shown in Fig. 1, lead to

$$\Gamma_2[q,G] = \frac{N\,\lambda_0}{4!\Gamma}\int_0^\tau dt\,G^2(t,t) + \frac{N\,g_0}{6!\Gamma}\int_0^\tau dt\,G^3(t,t) + \frac{3\,g_0}{6!\Gamma}\int_0^\tau dt\,q^2(t)\,G^2(t,t), \tag{27}$$

where $q(t) = \langle \phi_1(t)\rangle$ assuming that the symmetry is broken along the direction 1.

Stationarity of the functional $\Gamma[q,G]$ with respect to $q(t)$ and $G(t,t')$ yields the dynamical equations for the order parameter and its fluctuations, which read, respectively,

$$\left[-\frac{\partial^2}{\partial t^2} + m^2 + \frac{\lambda_0}{3!N}\,q^2(t) + \frac{g_0}{5!N^2}\,q^4(t) + \frac{\lambda_0}{3!}\,G(t,t) + \frac{g_0}{5!}\,G^2(t,t) + \frac{2\,g_0}{5!N}\,q^2(t)\,G(t,t)\right] q(t) = 0 \tag{28}$$

and

$$\left[-\frac{\partial^2}{\partial t^2} + m^2 + \frac{\lambda_0}{3!N}\,q^2(t) + \frac{g_0}{5!N^2}\,q^4(t) + \frac{\lambda_0}{3!}\,G(t,t) + \frac{g_0}{5!}\,G^2(t,t) + \frac{2\,g_0}{5!N}\,q^2(t)\,G(t,t)\right] G(t,t') = \Gamma\,\delta(t-t'). \tag{29}$$

These coupled dynamical equations are exact to leading order in N.

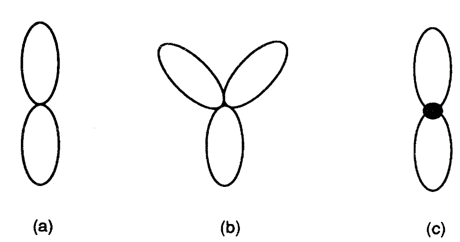

FIG. 1. Leading order 2PI diagrams.

It is interesting to note that Eqs. (28) and (29) can be obtained directly by introducing a quasilinearization scheme in the stochastic equation (20). A simple way of doing that is to average Eq. (20) and factorize the averages of the field using the Wick theorem. The quasilinearization procedure is somewhat *ad hoc* and difficult to improve, whereas in the formalism presented here successive corrections can be included systematically. We stress that in the present approach no *a priori* assumptions are required in order to obtain Eqs. (28) and (29). Thus it can also be seen as a justification of the factorization procedure employed in the quasilinear scheme.

Before discussing the solution of Eqs. (28) and (29), we note that these equations can also be derived by a variational approach to the path integral (8), or Hartree approximation, where one seeks for the best quadratic approximation for the action. The advantage of the variational approach is that it does not require any factorization of the averages. However, similar to the quasilinear approach, it is difficult to include corrections. For more details see Sec. VI.

IV. SOLUTION

We sketch briefly the solution of coupled effective dynamical equations (28) and (29), which can be cast into the form

$$\frac{\partial^2 q(t)}{\partial t^2} = F\big(q(t), G(t,t)\big)\, q(t), \tag{30}$$

$$\frac{\partial^2 G(t,t')}{\partial t^2} = F\big(q(t), G(t,t)\big)\, G(t,t') - \Gamma\delta(t-t'), \tag{31}$$

where

$$\begin{aligned} F\big(q(t), G(t,t)\big) = m^2 &+ \frac{\lambda_0}{3!N}\, q^2(t) + \frac{g_0}{5!N^2}\, q^4(t) \\ &+ \frac{\lambda_0}{3!}\, G(t,t) + \frac{g_0}{5!}\, G^2(t,t) \\ &+ \frac{2\,g_0}{5!N}\, q^2(t)\, G(t,t). \end{aligned} \tag{32}$$

We will eventually be interested in solutions for $\tau \to \infty$. Under this assumption the effective dynamical equations (30) and (31) can be reduced to simpler first-order nonlinear differential equations by using the following representation for $q(t)$ and $G(t,t')$:

$$q(t) = q(0)\, f_1(t) \tag{33}$$

and

$$G(t,t') = f_1(t')\, f_2(t)\, \theta(t'-t) + f_1(t)\, f_2(t')\, \theta(t-t') \tag{34}$$

with

$$f_1(t) = \exp\left(-\int_0^t dt'\, R(t')\right), \tag{35}$$

$$f_2(t) = \Gamma \exp\left[-\int_0^t dt'\, R(t')\right] \times \int_0^t dt' \exp\left[2\int_0^{t'} dt''\, R(t'')\right], \tag{36}$$

where the function $R(t)$ is solution of the first-order nonlinear differential equation

$$\frac{\partial R(t)}{\partial t} = R^2(t) - F\big(q(t), G(t,t)\big). \tag{37}$$

By inspection of Eqs. (32) and (37) it follows that $R(t)$ should have the functional form

$$R(t) = \alpha C(t) + \beta q^2(t) + \gamma, \tag{38}$$

where $C(t) = \lim_{t\to t'} G(t,t')$. The parameters α, β, and γ are determined by substituting $R(t)$ from Eq. (38) into Eq. (37) and eliminating $dC(t)/dt$ and $dq(t)/dt$ with the help of

$$\frac{\partial q(t)}{\partial t} = -R(t)q(t), \tag{39}$$

$$\frac{\partial C(t)}{\partial t} = -2R(t)C(t) + \Gamma, \tag{40}$$

obtained from Eqs. (33)–(36). We obtain the following solution for $R(t)$:

$$R(t) = \frac{\lambda}{6} C(t) + \frac{\lambda}{6N} q^2(t) + a. \tag{41}$$

In principle there exists another set of parameters, but it leads to an asymptotically unstable solution.

The first-order nonlinear differential equations (39)–(41) give the full description of the model (20) and (21) in the limit $N \to \infty$ for all times t. If $q(t)$ is not identically equal to zero, it is not straightforward to solve analytically the set of equations (39)–(41). Nevertheless these can easily be solved numerically for any set of initial conditions. We note that this is not the case for Eqs. (28) and (29). These indeed suffer of strong numerical instability and one has to devise a clever algorithm to handle them.

From the structure of the equations one can see that the O(N) symmetry dictates that the expectation value of the field $q(t)$ vanishes identically at all times if we assume the initial condition $q(0) = 0$. In this case the equation for $G(t,t')$ can be solved in closed analytical form. In fact, if we take the initial condition $C(t=0) = 0$, the solution reads

$$f_1(t) = e^{-at/2} \sqrt{\frac{1-(\alpha/\beta)}{e^{\lambda\Delta t/3} - (\alpha/\beta)\, e^{-\lambda\Delta t/3}}}, \tag{42}$$

$$f_2(t) = \frac{3\Gamma}{\lambda\Delta}\, e^{at/2} \sqrt{\frac{1-(\alpha/\beta)}{e^{\lambda\Delta t/3} - (\alpha/\beta)\, e^{-\lambda\Delta t/3}}} \times \sinh(\lambda\Delta t/3), \tag{43}$$

where

$$\alpha = \gamma + \Delta, \qquad \beta = \gamma - \Delta \tag{44}$$

with

$$\gamma = -\frac{3\,a}{\lambda}, \qquad \Delta = \sqrt{\gamma^2 + \frac{3\,\Gamma}{\lambda}}, \tag{45}$$

and a and λ are the coefficients entering in Eq. (21). Substitution of Eqs. (42)–(44) into Eq. (34) leads to the solution for $G(t,t')$. In the limit $t' \to t$ we recover the result of Ref. [9] for $C(t)$.

V. BEYOND THE HARTREE APPROXIMATION

The next leading terms of order $1/N$ can be included systematically by evaluating diagrams not included in Eq. (27). In the case $q(t) = 0$ the work is simplified since one does not have to consider separately transverse and parallel components of the correlation function $\langle \phi_i(t)\, \phi_j(t')\rangle_c$.

The diagrams contributing to the first corrections to Γ_2 are shown in Figs. 1(a), 1(b), and 2 and yield

$$\begin{aligned}\Gamma_2[G] = \frac{2}{4!\Gamma}\lambda_0 \int dt\, G^2(t,t) + \frac{6}{6!\Gamma} g_0 \int dt\, G^3(t,t) \\ -\frac{\lambda_0^2}{12^2\Gamma}\int dt \int dt'\, G^4(t,t') \\ -\frac{36}{(6!)^2\Gamma} g_0^2 \int dt \int dt'\, G^4(t,t')\, G(t,t)\, G(t',t') \\ -\frac{\lambda_0\, g_0^2}{720\Gamma}\int dt \int dt'\, G^4(t,t')\, G(t,t).\end{aligned} \quad (46)$$

To systematically improve this result and the Hartree approximation, one has to consider an infinite series of diagrams. A complete summation of the series (see Figs. 3 and 4) can be performed; however, only in the case of a system at equilibrium where time translational invariance holds. The final result for Γ_2 is valid to all orders in λ and to first order in $1/N$ and reads [14,15]

$$\begin{aligned}\lim_{\tau\to\infty} \Gamma_2[G]/\tau = \frac{N+2}{4!}\lambda_0 \int \frac{d\omega_1}{2\pi}\int \frac{d\omega_2}{2\pi}\widetilde{G}(\omega_1)\widetilde{G}(\omega_2) + \frac{N+6}{6!} g_0 \int \frac{d\omega_1}{2\pi}\int \frac{d\omega_2}{2\pi}\int \frac{d\omega_3}{2\pi}\widetilde{G}(\omega_1)\widetilde{G}(\omega_2)\widetilde{G}(\omega_3) \\ +\frac{1}{2}\int \frac{d\omega}{2\pi}\ln\left[1 + \left(\frac{\lambda_0}{3!} + \frac{g_0}{60}\, G(0,0)\right)\widetilde{\Pi}(\omega)\right] - \frac{1}{2}\int \frac{d\omega}{2\pi}\left(\frac{\lambda_0}{3!} + \frac{g_0}{60}\, G(0,0)\right)\widetilde{\Pi}(\omega),\end{aligned} \quad (47)$$

where $\widetilde{\Pi}(\omega)$ is the so-called vacuum polarization propagator

$$\widetilde{\Pi}(\omega) = \int \frac{d\omega}{2\pi}\, \widetilde{G}(\eta)\, \widetilde{G}(\eta+\omega). \quad (48)$$

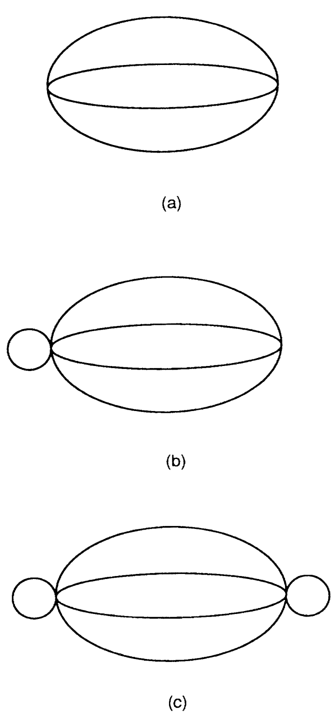

FIG. 2. 2PI diagrams contribution to the first $1/N$ corrections.

Upon differentiating with respect to $\widetilde{G}(\omega)$ the functional $\Gamma[G]$ with the $1/N$ corrections included one obtains [13]

$$\begin{aligned}\widetilde{G}^{-1}(\omega) = \omega^2 + m^2 + \frac{\lambda_0}{3!}\, G(0,0) + \frac{g_0}{5!}\, G^2(0,0) \\ +\frac{1}{N}\int \frac{d\omega_1}{2\pi}\frac{1}{\left\{1 + \left[\frac{\lambda_0}{3!} + \frac{g_0}{60}\, G(0,0)\right]\widetilde{\Pi}(\omega_1)\right\}} \\ \times\left[\left(\frac{\lambda_0}{3} + \frac{g_0}{30}\, G(0,0)\right)\widetilde{G}(\omega-\omega_1) + \widetilde{\Pi}(\omega_1)\right].\end{aligned} \quad (49)$$

Equation (49) represents the spectrum of the equilibrium fluctuations correct to order $1/N$. The study of these corrections will be the subject of a future paper.

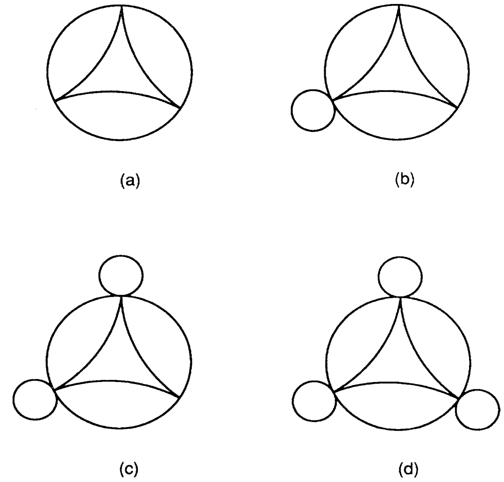

FIG. 3. Three-vertex 2PI diagrams contributing to the first $1/N$ corrections.

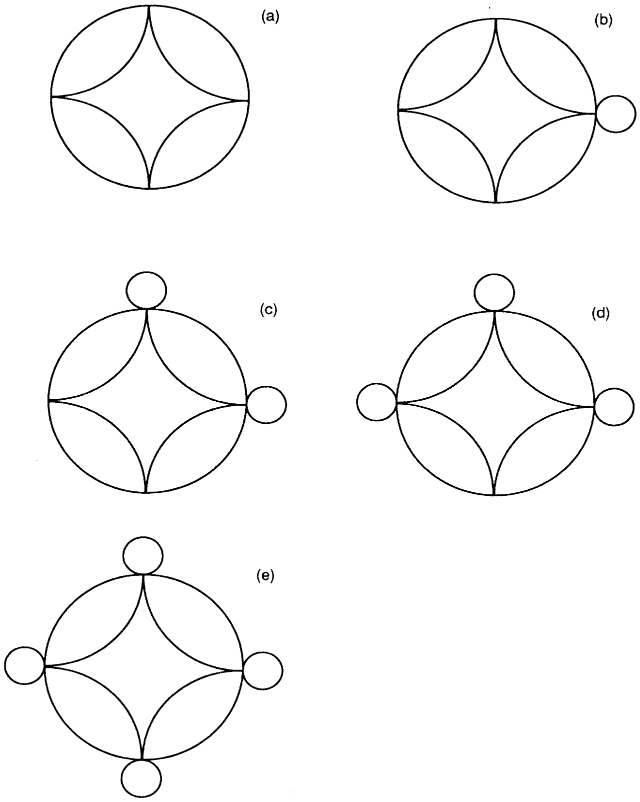

FIG. 4. Four-vertex 2PI diagrams contributing to the first $1/N$ corrections.

VI. VARIATIONAL APPROACH

The formalism discussed requires the presence of a small parameter to develop a perturbative expansion, e.g., the quantity $1/N$ in the above example. However, sometimes it is not always simple to identify such a parameter. The path integral formulation for the Langevin equation used here is amenable for an elegant variational approach, which is useful to obtain approximate equations even in the absence of small parameters.

By using the convexity of the exponential function we can use the Peierls-Feynman-Bogolubov inequality, which in our case reads

$$-\ln Z \leq -\ln Z_0 + \langle \mathcal{L} - \mathcal{L}_0 \rangle_0 \equiv W^{(1)}, \tag{50}$$

where $\mathcal{L}$ is action density given by

$$I = \int_0^\infty dt\, \mathcal{L} \tag{51}$$

and $\mathcal{L}_0$ is an arbitrary action density. The arbitrary parameter τ has been set to infinity. The average $\langle\,\rangle_0$ in (50) is done with respect to the probability distribution corresponding to $\mathcal{L}_0$ and Z_0 is the partition function associated with $\mathcal{L}_0$. As widely used in equilibrium statistical mechanics, we can assume the most general quadratic functional form for $\mathcal{L}_0$, i.e.,

$$\int_0^\infty dt\, \mathcal{L}_0 = \frac{1}{2\Gamma} \int_0^\infty dt \int_0^\infty dt' \sum_{i,j} \phi_i(t)\, u_{ij}(t,t')\, \phi_j(t') - \int_0^\infty dt \sum_i v_i(t)\, \phi_i(t), \tag{52}$$

where u_{ij} is a positive definite kernel. The Hartree method consists of choosing the arbitrary quadratic action (52), which minimizes the right-hand side of Eq. (50). To the best of our knowledge, this method does not seem to have been used in the theory of nonequilibrium processes so far.

Recalling that

$$\begin{aligned} W^{(0)}[u,v] &\equiv -\ln Z_0[u,v] \\ &= -\frac{1}{2} \ln ||u^{-1}(t,t')|| \\ &\quad -\frac{\Gamma}{2} \int_0^\infty dt \int_0^\infty dt' v(t)\, u^{-1}(t,t')\, v(t') \end{aligned} \tag{53}$$

and taking the derivatives of $W^{(0)}$ with respect to $v(t)$ we find

$$\frac{\delta W^{(0)}[u,v]}{\delta v(t)} = -q(t) = -\Gamma \int_0^\infty dt' u^{-1}(t,t')\, v(t'), \tag{54}$$

$$\frac{\delta^2 W^{(0)}[u,v]}{\delta v(t)\, \delta v(t')} = -G(t,t') = -\Gamma\, u^{-1}(t,t'), \tag{55}$$

which can be inverted to give

$$\int_0^\infty dt'\, u(t,t')\, q(t') = \Gamma\, v(t), \tag{56}$$

$$\int_0^\infty dt'\, u(t,t')\, G(t',t'') = \Gamma\, \delta(t-t''). \tag{57}$$

The arbitrary functions u_{ij} and v_i are determined by looking for the minimum of $W^{(1)}$ to be the best estimate of $\ln Z$.

For the model discussed above, by applying this variational method to (21) one finds

$$\begin{aligned} u(t,t') = \Bigg[&-\frac{\partial^2}{\partial t^2} + m^2 + \frac{\lambda_0}{3!N}\, q^2(t) + \frac{g_0}{5!N^2}\, q^4(t) \\ &+ \frac{\lambda_0}{3!}\, G(t,t) + \frac{g_0}{5!}\, G^2(t,t) \\ &+ \frac{2\, g_0}{5!N}\, q^2(t)\, G(t,t) \Bigg]\, \delta(t-t') \end{aligned} \tag{58}$$

and

$$v(t) = 0. \tag{59}$$

Inserting (58) and (59) in Eqs. (56) and (57) we find the same result as Eqs. (28) and (29).

To conclude the discussion of the Hartree method it is interesting to analyze what happens in the case $N = 1$. The variational nature in fact justifies its application even when the problem under scrutiny does not contain a natural small parameter around which to perform some sort of expansion.

Let us consider the case of a Langevin equation for a scalar field with cubic nonlinearity. The equations of motion for the average value of the field and for the fluctuations are [16]

$$\left[-\frac{\partial^2}{\partial t^2} + m^2 + \frac{\lambda_0}{3!}\,q^2(t) + \frac{g_0}{5!}\,q^4(t) + \frac{\lambda_0}{2}\,G(t,t) + \frac{g_0}{8}\,G^2(t,t) + \frac{g_0}{12}\,q^2(t)\,G(t,t)\right] q(t) = 0, \quad (60)$$

$$\left[-\frac{\partial^2}{\partial t^2} + m^2 + \frac{\lambda_0}{2}\,q^2(t) + \frac{g_0}{4!}\,q^4(t) + \frac{\lambda_0}{2}\,G(t,t) + \frac{g_0}{8}\,G^2(t,t) + \frac{g_0}{8}\,q^2(t)\,G(t,t)\right] G(t,t') = \Gamma\,\delta(t-t'). \quad (61)$$

With the substitution

$$q(t) = \exp\left(-\int_0^t d\tau R_q(\tau)\right) \quad (62)$$

and $G(t,t')$ given by Eq. (34) we obtain two equations for $R_q(t)$ and $R(t)$ analogous to Eq. (37). Using the trial solution

$$R_q(t) = \alpha_q\,C(t) + \beta_q\,q^2(t) + \gamma_q \quad (63)$$

and $R(t)$ given by Eq. (38) we find, up to terms quadratic in the coupling constant λ, $\alpha_q = \alpha = \lambda/2$, $\beta_q = \lambda/6$, $\beta = \lambda/2$, and $\gamma_q = \gamma = a$. The value of the coefficients coincides with the value obtained in the so-called Langer–Bar-on–Miller approximation [17] to the Langevin equation, a result that was also rediscovered few years ago [5] on the basis of a somewhat *ad hoc* variational principle. We believe that the present derivation, being based on a path integral formulation of the stochastic equations, makes the underlying physical assumptions more clear.

VII. CONCLUSIONS

Many problems arising in the study of physical problems are most naturally represented in terms of systems of nonlinear stochastic differential equations. Several approximation schemes have been developed to treat the nonlinear aspect of equations. Usually they are based on some reasonable assumptions. The advantage of these approaches is that they may lead to relatively simple equations. The drawback is that it is not simple to improve the quality of the approximation. In this paper we have presented an alternative approach to the study of nonlinear Langevin equation that allows for systematic development of approximation schemes. The basic idea is to reduce the nonlinear Langevin equation to an equivalent equilibrium problem to which the methods of conventional field theory can be applied. A particular well suited perturbative scheme is that developed in quantum field theory by Cornwall, Jackiw, and Tomboulis [6]. The major advantage is that it leads to a variational principle for the physical quantities of interest.

The method is illustrated by applying it to an N-component Ginzburg-Landau equation. The leading contributions for $N \to \infty$ reproduces the known equations obtained with other methods, e.g., stochastic linearization. By means of the method proposed here we are able to evaluate the next order corrections for the order parameter $q(t)$ and two-time connected correlation function $G(t,t')$.

The study of these is, however, more involved and has not been included in this paper. This will be part of future work. It will also be of interest to extend the present approach to higher dimensions and to explore numerically the predictions of the present approach to finite values of N.

Finally, we present a variational approach for the dynamical Hartree approximation based on the path integral formulation presented here. The advantage of this is that it can be used even in the absence of a small parameter to set up a perturbative expansion.

The two methods presented in Secs. III and VI are both variational, but they are different in their spirit. The Cornwall-Jackiw-Tomboulis method does not assume a specific form of the action and the dynamical equations resulting from the variation of the effective potential $\Gamma[q,G]$ with respect to the local and bilocal sources are in principle exact. The explicit form of $\Gamma[q,G]$ has to be calculated by means of a physical approximation. Within the Hartree dynamical approximation, on the other hand, one imposes from the beginning a quadratic form for the action. After optimizing it with respect to the variational parameters one obtains self-consistent dynamical equations, similar to the dynamical equations derived, e.g., in the quasilinearization scheme.

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